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LETTER TO THE EDITOR

Boundary terms in the Schwinger–DeWitt expansion: flat space results

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Abstract. Asymptotic expansions for the Green function of the diffusion equation are obtained for both hyperspherical and arbitrarily shaped smooth boundaries in a flat embedding space.

1. Introduction

The role of boundary contributions to the asymptotic expansion of the integrated kernel of the diffusion equation, $K(t)$, for a manifold $\mathcal{M}$ with boundary $\partial\mathcal{M}$ has recently been emphasised in separate contexts (Christensen and Duff 1978, Dowker and Kennedy 1978). The present interest in this expansion lies in the domain of quantum field theory where it is commonly known as the (integrated) Schwinger–DeWitt proper-time expansion, obtained from $K(t)$ by rotating t to $\mathrm{i}\tau$, τ being the proper-time parameter. In this Letter we give the first few terms in the $K(t)$ expansion for the special case of higher-dimensional spheres, and also obtain dimension-independent results for the first few coefficients in the expansion valid for an arbitrary smooth boundary in a flat embedding space. Dirichlet boundary conditions are considered throughout.

2. General expressions for the boundary terms

The Green function for the diffusion equation $K(x, x'; t)$ on a $(p+2)$-dimensional manifold with positive definite metric $g_{\mu\nu}$ is defined by

$$\left(\frac{\partial}{\partial t}-\Box\right)K(x, x'; t)=\delta(t)\delta(x, x') \tag{1}$$

where

$$\Box=\frac{1}{\sqrt{g}}\partial_\mu\,(\sqrt{g}\,g^{\mu\nu}\partial_\nu)$$

is the $(p+2)$-dimensional Laplace–Beltrami operator and $\delta(x, x')$ is the covariant δ function. The solution to (1) depends of course on the boundary conditions employed but, as has been shown by Greiner (1971), the integrated Green function

$$K(t)=\int_{\mathcal{M}} K(x,x;t)\,\mathrm{d}V, \tag{2}$$

where $\mathrm{d}V$ is the volume form on $\mathcal{M}$, has the asymptotic expansion as $t\to 0$

$$K(t)=\frac{1}{(4\pi t)^{\frac{1}{2}p+1}}\sum_{l=0,\frac{1}{2},1,\ldots} c_l t^l+\text{ES}, \tag{3}$$

ES representing terms exponentially small as $t\to 0$. The c_l in this expansion may be written (Greiner 1971) as a volume part plus a boundary part

$$c_l=\int_{\mathcal{M}} a_l(x,x)\,\mathrm{d}V+\int_{\partial\mathcal{M}} b_l(x)\,\mathrm{d}\sigma\equiv a_l+b_l, \tag{4}$$

the $a_l(x,x')$ being the usual Minakshisundaram coefficients for the manifold without boundary, and $\mathrm{d}\sigma$ the volume form on $\partial\mathcal{M}$. Further comments on these coefficients may be found in Dowker and Kennedy (1978), but here we concentrate our attention on the boundary contributions in (4).

Since $K(t)$ is a dimensionless scalar and the b_l must take the form of integrals over $\partial\mathcal{M}$ of certain invariants on $\partial\mathcal{M}$, we can immediately deduce that $b_0=0$. On restricting our discussion to the case of a flat embedding space we can, for $l\geqslant\frac{1}{2}$, construct invariants using the second fundamental form, K_{ij}, on $\partial\mathcal{M}$ and the intrinsic curvature, R_{ij}, of $\partial\mathcal{M}$ to give the general expressions

$$b_{1/2}=\alpha_{1/2}\int_{\partial\mathcal{M}}\mathrm{d}\sigma \tag{5a}$$

$$b_1=\alpha_1\int_{\partial\mathcal{M}}(\operatorname{tr}K)\,\mathrm{d}\sigma \tag{5b}$$

$$b_{3/2}=\int_{\partial\mathcal{M}}[\alpha_{3/2}(\operatorname{tr}K)^2+\beta_{3/2}(\operatorname{tr}K^2)+\gamma_{3/2}R]\,\mathrm{d}\sigma \tag{5c}$$

$$b_2=\int_{\partial\mathcal{M}}[\alpha_2(\operatorname{tr}K)^3+\beta_2(\operatorname{tr}K^3)+\gamma_2(\operatorname{tr}K)(\operatorname{tr}K^2)+\delta_2R(\operatorname{tr}K)+\epsilon_2R_{ij}K^{ij}]\,\mathrm{d}\sigma \tag{5d}$$

where $R=R_i^{\ i}$ and $\operatorname{tr}K=K^i_{\ i}$. The relations

$$R=(\operatorname{tr}K)^2-(\operatorname{tr}K^2) \tag{6a}$$

$$R_{ij}K^{ij}=(\operatorname{tr}K)(\operatorname{tr}K^2)-(\operatorname{tr}K^3), \tag{6b}$$

valid for a flat $\mathcal{M}$ (Eisenhart 1926), reduce equations (5c), (5d) to

$$b_{3/2}=\int_{\partial\mathcal{M}}[\alpha'_{3/2}(\operatorname{tr}K)^2+\beta'_{3/2}(\operatorname{tr}K^2)]\,\mathrm{d}\sigma \tag{5c$'$}$$

$$b_2=\int_{\partial\mathcal{M}}[\alpha'_2(\operatorname{tr}K)^3+\beta'_2(\operatorname{tr}K^3)+\gamma'_2(\operatorname{tr}K)(\operatorname{tr}K^2)]\,\mathrm{d}\sigma. \tag{5d$'$}$$

For a flat $\mathcal{M}$ we also have $a_l = 0$, $l \neq 0$, and $a_0 = |\mathcal{M}|$, the volume of $\mathcal{M}$. From (3) and (4) the $K(t)$ expansion then takes the form

$$K(t) = \frac{|\mathcal{M}|}{(4\pi t)^{\frac{1}{2}p+1}} + K'(t) \tag{7a}$$

$$K'(t) = \frac{1}{(4\pi t)^{\frac{1}{2}p+1}} \sum_{l=\frac{1}{2},1,\frac{3}{2},\ldots} b_l t^l + \text{ES}. \tag{7b}$$

The object would now be to fix the coefficients in equations (5a), (5b), (5c'), and (5d') by calculating $K(t)$ either for explicitly soluble examples or directly for an arbitrarily shaped boundary. Both forms of approach are taken in the following two sections.

3. Results for higher-dimensional spheres

In this section we draw on the work of Stewartson and Waechter (1971) and of Waechter (1972). Taking the Laplace transform of $K(t)$

$$\bar{K}(s^2) = \int_0^\infty e^{-s^2 t} K(t)\, dt \tag{8}$$

and using the properties of hyperspherical harmonics (Magnus and Oberhettinger 1948, Erdélyi 1953), the usual spherical harmonics being included as a special case, we find, omitting the details, for S^{p+1} of radius a bounding R^{p+2}

$$\bar{K}'(s^2) = -a^2 \sum_{m=0}^\infty \frac{(m+\frac{1}{2}p)(m+p-1)!}{p!\,m!} f(m+\tfrac{1}{2}p; sa), \tag{9}$$

the prime denoting the separating off of the Weyl term as in (7a) and $f(m+\frac{1}{2}p; sa)$ being the same combination of Bessel functions as encountered by Stewartson and Waechter (1971),

$$f(\nu; sa) = \left(1 + \frac{\nu^2}{s^2 a^2}\right) I_\nu(sa) K_\nu(sa) - I'_\nu(sa) K'_\nu(sa) - \frac{I'_\nu(sa)}{sa I_\nu(sa)}. \tag{10}$$

An expansion for $f(\nu; sa)$ valid for large s is given in Stewartson and Waechter (1971), and is used in (9) after replacement of the sum by a contour integration to give an asymptotic expansion for $\bar{K}'(s^2)$. Taking the inverse Laplace transform then yields, on neglecting terms exponentially small as $t \to 0$,

$p = 1$
$$K'(t) = -\frac{a^2}{4t} + \frac{a}{3\pi^{1/2} t^{1/2}} - \frac{1}{48} - \frac{2t^{1/2}}{315\pi^{1/2} a} - \frac{t}{960 a^2} + O(t^{3/2}) \tag{11a}$$

$p = 2$
$$K'(t) = -\frac{\pi^{1/2} a^3}{16 t^{3/2}} + \frac{a^2}{8t} - \frac{11\pi^{1/2} a}{512 t^{1/2}} - \frac{1}{180} - \frac{35\pi^{1/2} t^{1/2}}{2^{16} a} + O(t) \tag{11b}$$

$p = 3$
$$K'(t) = -\frac{a^4}{24 t^2} + \frac{a^3}{9\pi^{1/2} t^{3/2}} - \frac{a^2}{32 t} - \frac{4a}{945\pi^{1/2} t^{1/2}} + \frac{17}{11520} + O(t^{1/2}) \tag{11c}$$

$p = 4$
$$K'(t) = -\frac{\pi^{1/2} a^5}{2^7 t^{5/2}} + \frac{5a^4}{192 t^2} - \frac{125\pi^{1/2} a^3}{3 \,.\, 2^{12} t^{3/2}} + \frac{2159\pi^{1/2} a}{3 \,.\, 2^{19} t^{1/2}} + O(1). \tag{11d}$$

We note in particular that, from (7b) and (11d), $b_2 = 0$ for an S^5 boundary in R^6.

Although the calculation of (11*a*) essentially follows the same treatment as Waechter (1972) we obtain different results for $b_{3/2}$, b_2, and $b_{5/2}$, in particular we do not find $b_{3/2}=0$ for an S^2 boundary. The source of the discrepancy lies in a contribution from a series of exponentials in the expansion of tan $\pi\nu$, a factor which appears in the conversion of the summation in (9) to a contour integration for the $p=1$ case.

The results in equations (11) could be used to fix the coefficients in § 2 but we prefer to do this by a more direct approach in the following section.

4. Expansion for an arbitrary smooth boundary

We again draw on the work of Stewartson and Waechter (1971) and more especially Waechter (1972) in this section.

Let $(x, y, \ldots, z)$ be a $(p+2)$-dimensional Cartesian coordinate system, the $x=0$ hypersurface being the tangent plane at some point P on $\partial\mathcal{M}$ with coordinates $(0, y_0, \ldots, z_0)$. The equation of the boundary $\partial\mathcal{M}$ can then be expanded as a Taylor series about P

$$x=\alpha_{ij}(q^i-q_0^i)(q^j-q_0^j)+\alpha_{ijk}(q^i-q_0^i)(q^j-q_0^j)(q^k-q_0^k) \\ +\alpha_{ijkl}(q^i-q_0^i)(q^j-q_0^j)(q^k-q_0^k)(q^l-q_0^l)+\mathrm{O}[(q-q_0)^5] \tag{12}$$

where the α coefficients are expressible in terms of the second fundamental form and q^i is one of $(y, \ldots, z)$.

Defining $\bar{K}^{(\nu)}(x, x'; s^2)$ through

$$\bar{K}^{(\nu)}(x, x'; s^2)=\int_0^\infty \mathrm{e}^{-s^2t}t^{\nu-1}K(x, x'; t)\,\mathrm{d}t \tag{13}$$

and using the expression for $K(x, x'; t)$ for the manifold without boundary

$$K_\infty(x, x'; t)=\frac{1}{(4\pi t)^{\frac{1}{2}p+1}}\exp[-(R_{x-x',y-y',\ldots,z-z'})^2/4t] \tag{14}$$

where

$$R_{x-x',y-y',\ldots,z-z'}=[(x-x')^2+(y-y')^2+\ldots+(z-z')^2]^{1/2},$$

we can write

$$\bar{K}^{(\nu)}\left(\frac{\xi}{s},\frac{\xi_0}{s};s^2\right) \\ =\frac{s^{p-2\nu+2}}{2^{p/2+\nu}\pi^{p/2+1}}\left(\frac{K_{(p/2-\nu+1)}(R_{\xi-\xi_0,\eta-\eta_0,\ldots,\mu-\mu_0})}{(R_{\xi-\xi_0,\eta-\eta_0,\ldots,\mu-\mu_0})^{\frac{1}{2}p-\nu+1}} \right. \\ -\left.\frac{K_{(p/2-\nu+1)}(R_{\xi+\xi_0,\eta-\eta_0,\ldots,\mu-\mu_0})}{(R_{\xi+\xi_0,\eta-\eta_0,\ldots,\mu-\mu_0})^{\frac{1}{2}p-\nu+1}}\right) \\ -\int_{-\infty}^\infty \mathrm{d}\eta_2\int_{-\infty}^\infty \mathrm{d}\zeta_2\ldots\int_{-\infty}^\infty \mathrm{d}\mu_2\, f(\eta_2,\zeta_2,\ldots,\mu_2;s) \\ \times\frac{\partial}{\partial\xi}\left(\frac{K_{(p/2-\nu+1)}(R_{\xi,\eta-\eta_0-\eta_2,\ldots,\mu-\mu_0-\mu_2})}{(R_{\xi,\eta-\eta_0-\eta_2,\ldots,\mu-\mu_0-\mu_2})^{\frac{1}{2}p-\nu+1}}\right). \tag{15}$$

In (15) we have used the notation $\bar{\xi} \equiv (\xi, \eta, \ldots, \mu)$, where the $(\xi, \eta, \ldots, \mu)$ coordinates are related to the $(x, y, \ldots, z)$ coordinates through

$$(\xi, \eta, \ldots, \mu) \equiv s(x, y, \ldots, z), \tag{16}$$

and $K_\nu(z)$ is the modified Bessel function of the second kind. The form of equation (15) is an extension of the usual method of images to include, besides the image charge at $(-\xi_0, \eta_0, \ldots, \mu_0)$, a dipole distribution of density $f(\eta, \zeta, \ldots, \mu; s)$ over the $\xi = 0$ hypersurface.

The introduction of the $t^{\nu-1}$ factor in (13) is necessitated by the fact that, since $K(x, x'; t)$ contains inverse powers of t as in equation (14), the formal Laplace transform (13) with $\nu = 1$ is undefined. When working with the Laplace transform parameter s^2 this problem manifests itself in the form of divergent integrals accompanying those terms which correspond, under inversion of the Laplace transform, to inverse powers of t. The $t^{\nu-1}$ factor is similar to that employed in zeta function regularisation (Dowker and Critchley 1976): we assume Re $\nu > \frac{1}{2}p + 1$ until the inverse Laplace transform has been taken and then let $\nu \to 1$, the divergences neatly cancelling out.

The scaled coordinates (16) may be substituted in (12) to obtain the equation of the boundary as a power series in $1/s$. The $f(\eta, \zeta, \ldots, \mu; s)$ is also written as a power series in s, the requirement that (15) vanish on $\partial\mathcal{M}$ giving the coefficients in this series. Inserting this information back in (15), setting $\bar{\xi} = \bar{\xi}_0$ and integrating over $\mathcal{M}$, taking the inverse Laplace transform and finally letting $\nu \to 1$ yields a $K(t)$ expansion in the form of equations (7) with dimension-independent coefficients $b_{1/2}, b_1, b_{3/2}, b_2$. Omitting the details, we find after some heavy algebra:

$$b_{1/2} = -\frac{\pi^{1/2}}{2} |\partial\mathcal{M}| \tag{17a}$$

$$b_1 = \frac{1}{3} \int_{\partial\mathcal{M}} (\mathrm{tr}\, K)\, \mathrm{d}\sigma \tag{17b}$$

$$b_{3/2} = \frac{\pi^{1/2}}{192} \int_{\partial\mathcal{M}} [10(\mathrm{tr}\, K^2) - 7(\mathrm{tr}\, K)^2]\, \mathrm{d}\sigma \tag{17c}$$

$$b_2 = \frac{1}{945} \int_{\partial\mathcal{M}} [5(\mathrm{tr}\, K)^3 + 40(\mathrm{tr}\, K^3) - 33(\mathrm{tr}\, K)(\mathrm{tr}\, K^2)]\, \mathrm{d}\sigma. \tag{17d}$$

The results obtained by this direct approach are of the form suggested in § 2, and, as is easily verified from equations (7), the resultant $K(t)$ expansion agrees with all the higher-dimensional sphere results of equations (11) and also with the expansion for S^1 bounding R^2 given by Stewartson and Waechter (1971). Equations (17a) and (17b) agree with the more general curved space expansion of McKean and Singer (1967) but again we find disagreement with Waechter's (1972) result for $b_{3/2}$, proved by him for $p = 1$ and quoted by us in a previous paper (Dowker and Kennedy 1978).

5. Discussion

Although our treatment of boundary terms has so far been restricted to the case of a flat embedding space, where they find an application in the Casimir effect (Dowker and Kennedy 1978), our main interest lies with the form these terms take when $\mathcal{M}$ is curved with particular relevance to surface actions in quantum gravity (Christensen and Duff 1978). It appears that for this more general case the form for b_2 in equation (5d) could be augmented by additional invariants (the author thanks Stuart Dowker for discussions on this point). The present flat $\mathcal{M}$ results would then only give certain algebraic relations between the coefficients of the invariants. The curved space case will be examined in a further publication along with fuller details on the calculations of §§ 3 and 4.

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